Lecture 7: Pigeonhole principle and its variants

Main points:

- PHP, extended PHP, probabilistic PHP(twin paradox)
- PHP has **MANY** applications!

1 Different forms of Pigeonhole principle(PHP)

PHP, the standard form: if there are $n+1$ pigeons but there are only n holes, then at least two pigeons are in the same hole.

slightly general: if if there are m pigeons but there are only n holes, and $m > n$, then at least two pigeons are in the same hole.

PHP, the function form: Let $f : X \to Y$ be a function, and $|X| > |Y|$, then there must be a value $y \in Y$ for which there are at least two distinct $x_1 \neq x_2$ satisfying $f(x_1) = f(x_2) = y$.

extended PHP: if there are $nk+1$ pigeons but there are only n holes, then at least $k+1$ pigeons are in the same hole.

extended PHP: Let $f : X \to Y$ be a function, then there must be a value $y \in Y$ for which there are at least $\left\lceil \frac{|X|}{|Y|} \right\rceil$ distinct x_i satisfying $f(x_i) = y$.

probabilistic PHP: PHP is used to guarantee that some phenomenon *must* happen when pigeons are more than holes, while probabilistic PHP means to guarantee some phenomenon will happen *with high probability (whp)* as long as pigeons are sufficently many compared to holes, even though pigeons are less than holes. Example: the twin paradox.

2 Applications

Example 1 (for fun statistics). Are there two people in Shenzhen with the same strands of hair?

(1) no one has more than $500,000$ strands of hair $\Longleftrightarrow$ number of holes = 500,001.

(2) there are 17 million people in Shenzhen. $\Longleftrightarrow$ number of pigeons = 17 million.

Since 17 million > 500,001, there must be at least two people having the same strands of hair.

In fact, by extended PHP, we know there must be at least $> 17,000,000/500,001$, which gives 34 people with the same strands of hair.

Example 2 (for training, or for geometry). 50 shots in a square of 70×70 square cm, there must be two shots closer than 15cm.

Solution: partition the square into $7 \times 7 = 49$ small squares.

Example 3 (for numbers). Choose any 13 numbers (or more) from $2, 3, \ldots, 40$, there must be two numbers a and b satisfying $(a, b) > 1$.

Note: there are $\binom{39}{13} = 8122425444$ different choices, and for each choice of 13 numbers, you can verify whethere there are two numbers satisfying the condition as required using at most $13^2 = 169$ comparisons, so, the total computational steps required is at most

$$\binom{39}{13} \times 13^2 = 8122425444 \times 169 = 1.372689900036 \times 10^{12}$$

This number, though pretty large, is still within the computational power of our computers. So, one can write a simple brute-force algorithm to check the statement. However, using PHP and some simple observation of numbers, we can save all the computation!

Solution: The prime numbers between $2, 3, ..., 40$ are:

$$P = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,$$

exactly 12 of them. Let $S \subseteq \{2, 3, ..., 40\}$ be the set you choose. Now, consider a function:

$$f : S \to P, \quad s \mapsto f(s)$$

where $f(s)$ is the least prime factor of s. Apply PHP to draw the conclusion.

Example 4 (for geometry). Among any 9 people on earth, there must be 6 people in the same hemisphere (including boundary of hemisphere)

Solution: any two people determine a big circle which divides the sphere into two parts.

Example 5 (for combinatorics). Show $R(3, 3) \leq 6$.

We've seen that one can prove this simply by doing all the case analysis. With PHP we can do it in a simpler way.

Proof. sketch: Fix one people A, look at A's friends and stranger relationship to the other 5 people, apply extended PHP, and deduce the conclusion from it. □

Example 6 (prob PHP). twin paradox. Read the corresponding section in reading material.

After class: reading material suggested.